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VIBRATION ANALYSIS OF BEAMS WITH A TWO DEGREE-OF-FREEDOM SPRING–MASS SYSTEM

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Abstract—In this paper, the natural frequencies and mode shapes of a Bernoulli–Euler beam with a two degree-of-freedom spring–mass system are determined by using Laplace transform with respect to the spatial variable. The deterministic and random vibration responses of the beam are obtained by using model analysis. The various spring constants of the boundary conditions of the beam are considered to model those special structures and machines, also different parameters of the spring–mass system are taken into consideration to study the fundamental natural frequencies of the beam. It should be emphasized that the proposed analysis is quite important for the design of some components of the buildings or machine tools. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Vibration problem of beams carrying elastically-mounted concentrated masses have been extensively investigated recently. They are usually used to design the components of buildings and some accessory components of machine structures and so on. The following researchers have already made many contributions to the development of this field. Laura *et al.* (1977) studied the dynamic behavior of the structure elements with elastically-mounted concentrated masses. Lau (1981) obtained the fundamental frequency of a constrained beam. Gurgoze (1984) performed the vibration analysis of the restrained beams and rods with point masses. Ercoli and Laura (1987) did both the analytical and experimental investigations on continuous beams carrying elastically-mounted masses. Jen and Magrab (1993) obtained an exact solution for the natural frequencies and mode shapes for a beam elastically constrained at its end and to which a rigid mass is elastically-mounted. Meanwhile, Rossi *et al.* (1993) investigated the free vibrations of Timoshenko beams carrying elastically-mounted concentrated masses. Kukla and Posiadala (1994) adopted the Green function method to get the closed form expressions for the natural frequencies of beams with elastically-mounted masses. Incidentally, Nicholson and Bergman (1986) studied the vibration of damped plate-oscillator systems. Avalos *et al.* (1993) investigated the free vibrations of a simply-supported plate carrying an elastically-mounted concentrated mass, also, Avalos *et al.* (1994) performed the free vibration analysis for a circular plate. Most of the above studies only dealt with the free vibrations, however, the present study considered the free, deterministic and random vibrations and this is probably the first paper where a forced vibration situation is treated on such systems.

In this study, we consider a beam with general boundary conditions, carrying a two degree-of-freedom spring–mass system to model the machine tools or structures on the beam. The free vibration solution is obtained by using the Laplace transform with respect to the spatial variable. This technique has been shown in the following papers (Magrab, 1968; Hamada, 1981; Chang, 1993). The forced vibration is solved by the eigenfunction expansion method which was also adopted and investigated by the researchers (Chang, 1993; Hull, 1994). The objective of the paper is to obtain the natural frequencies, mode shapes and deterministic and random vibration response of the beam, furthermore, the effect of changing the parameters of the system on the natural frequencies, mode shapes and forced vibration response are also studied.

2. GOVERNING EQUATION OF MOTION

The partial differential equation of a uniform beam with an attached two degree-of-freedom spring–mass system (Fig. 1), according to Bernoulli–Euler theory, is the well-known expression as follows

$$EI\frac{\partial^4 y_b}{\partial x^4}+C\frac{\partial y_b}{\partial t}+\rho A\frac{\partial^2 y_b}{\partial t^2}=F(x,t)+[K_1(y_1+d_1\theta-y_b)\,\delta(x-x_1)$$
$$+K_2(y_1-d_2\theta-y_b)\,\delta(x-x_2)]. \quad (1)$$

The equation of motion of the two degree-of-freedom spring–mass system can be written as follows

$$m_1\ddot{y}_1+K_1[y_{11}-y_b(x_1)]+K_2[y_{12}-y_b(x_2)]=0 \quad (2)$$

$$I_1\ddot{\theta}-K_2[y_{12}-y_b(x_2)]d_2+K_1[y_{11}-y_b(x_1)]d_1=0 \quad (3)$$

where $y_{11}=y_1+d_1\theta$, $y_{12}=y_1-d_2\theta$, $y_b(x,t)$ is the displacement of the beam, E is the modulus of elasticity of the beam, ρ is the density of the beam, x is the spatial location, t is the time, A is the cross-section area of the beam, $F(x,t)$ is the applied force per unit length, δ is the Dirac delta function, and K_1 and K_2 are spring constants. The boundary conditions of the system are obtained by balancing the forces and moments at both ends of the beam, which can be expressed as follows

$$EIy_b''(0,t)=\beta_1 y_b'(0,t) \quad (4)$$

$$EIy_b'''(0,t)=-K_3 y_b(0,t) \quad (5)$$

$$EIy_b''(L,t)=-\beta_2 y_b'(L,t) \quad (6)$$

$$EIy_b'''(L,t)=K_4 y_b(L,t) \quad (7)$$

where K_3 and K_4 are spring constants.

The initial condition of the system are assumed as follows

$$y_b(x,0)=0 \quad (8)$$

$$\dot{y}_b(x,0)=0. \quad (9)$$

When the two degree-of-freedom system is assumed to undergo harmonic oscillations, that is, $\ddot{y}_1=-\omega^2 y_1$, $\ddot{\theta}=-\omega^2\theta$, eqns (2) and (3) can be written as

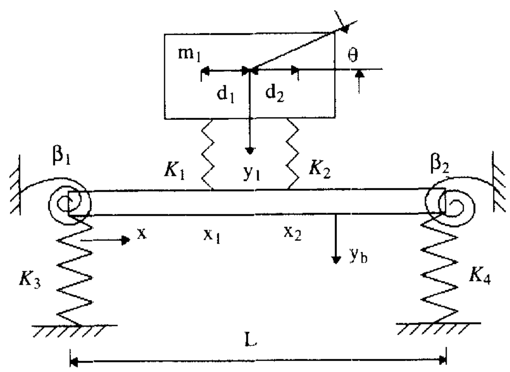

Fig. 1. Euler beam with a two degree-of-freedom system.

$$\begin{bmatrix} K_1+K_2-m_1\omega^2 & -K_2d_2+K_1d_1 \\ -K_2d_2+K_1d_1 & K_2d_2^2+K_1d_1^2-I_1\omega^2 \end{bmatrix}\begin{Bmatrix} y_1 \\ \theta \end{Bmatrix}=\begin{Bmatrix} K_1y_b(x_1)+K_2y_b(x_2) \\ -K_2d_2y_b(x_2)+K_1d_1y_b(x_1) \end{Bmatrix}. \quad (10)$$

For simplicity, eqn (10) can also be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{Bmatrix} y_1 \\ \theta \end{Bmatrix}=\begin{Bmatrix} K_1y_b(x_1)+K_2y_b(x_2) \\ -K_2d_2y_b(x_2)+K_1d_1y_b(x_1) \end{Bmatrix} \quad (11)$$

where

$$a_{11} = K_1+K_2-m_1\omega^2 \quad (12)$$

$$a_{12} = a_{21} = -K_2d_2+K_1d_1 \quad (13)$$

$$a_{22} = K_2d_2^2+K_1d_1^2-I_1\omega^2. \quad (14)$$

Solving eqns (2) and (3) for y_1 and θ gives the following:

$$y_1 = [(a_{22}K_1-a_{12}K_1d_1)y_b(x_1)+(a_{22}K_2+a_{12}K_2d_2)y_b(x_2)]/(a_{11}a_{22}-a_{12}^2) \quad (15)$$

$$\theta = [(a_{11}K_1d_1-a_{12}K_1)y_b(x_1)+(a_{11}K_2d_2+a_{12}K_2)y_b(x_2)]/(a_{11}a_{22}-a_{12}^2). \quad (16)$$

3. FREE VIBRATION ANALYSIS

Substituting eqns (15) and (16) into the right-hand side of the governing eqn (1) yields the following

$$EIy''''_b+C\dot{y}_b+\rho A\ddot{y}_b = F(x,t)+K_1B_1y_b(x,t)\,\delta(x-x_1)+K_2B_2y_b(x,t)\,\delta(x-x_2) \quad (17)$$

where

$$B_1 = (a_{22}K_1-2a_{12}K_1d_1+a_{11}K_1d_1^2)/(a_{11}a_{22}-a_{12}^2)-1 \quad (18)$$

$$B_2 = (a_{22}K_2+2a_{12}K_2d_2+a_{11}K_2d_2^2)/(a_{11}a_{22}-a_{12}^2)-1. \quad (19)$$

For the undamped free vibration analysis, $F(x,t)$, and damping coefficient, C, are considered as zeroes and $y_b = \phi(x)T(t)$ is assumed, then it is quite straightforward to derive the following equation:

$$\phi''''-\lambda^4\phi = [K_1B_1\phi(x)\,\delta(x-x_1)+K_2B_2\phi(x)\,\delta(x-x_2)]/EI \quad (20)$$

where

$$\lambda^4 = \rho A\omega^2/EI.$$

In order to solve eqn (20), we use the Laplace transform with respect to x, and solve it for the transformed $\bar{\phi}(s)$, then take its inverse to obtain the following

$$\begin{aligned}\phi(x) &= 0.5(\cosh\lambda x+\cos\lambda x)\phi(0)+0.5(\sinh\lambda x+\sin\lambda x)\phi'(0)/\lambda \\ &\quad +0.5(\cosh\lambda x-\cos\lambda x)\phi''(0)/\lambda^2+0.5(\sinh\lambda x-\sin\lambda x)\phi'''(0)/\lambda^3 \\ &\quad +\frac{0.5}{EI\lambda^3}\sum_{i=1}^{2}\phi(x_i)K_iB_i(\lambda)[\sinh\lambda(x-x_i)-\sin\lambda(x-x_i)]u(x-x_i) \quad (21)\end{aligned}$$

where $u(x)$ denotes the unit step function in eqn (21).

In order to simplify the expressions of eqn (21), we assume the following:

$$R_1(\lambda x) = 0.5(\cosh \lambda x + \cos \lambda x) \tag{22}$$

$$R_2(\lambda x) = 0.5(\sinh \lambda x + \sin \lambda x) \tag{23}$$

$$R_3(\lambda x) = 0.5(\cosh \lambda x - \cos \lambda x) \tag{24}$$

$$R_4(\lambda x) = 0.5(\sinh \lambda x - \sin \lambda x). \tag{25}$$

Then eqn (21) can be written as

$$\begin{aligned}\phi(x) &= R_1(\lambda x)\phi(0) + \frac{1}{\lambda}R_2(\lambda x)\phi'(0) + \frac{1}{\lambda^2}R_3(\lambda x)\phi''(0) \\ &\quad + \frac{1}{\lambda^3}R_4(\lambda x)\phi'''(0) + \frac{1}{EI\lambda^3}\sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)R_4[\lambda(x-x_i)]u(x-x_i).\end{aligned} \tag{26}$$

To solve for the four unknown quantities $\phi(0)$, $\phi'(0)$, $\phi''(0)$, $\phi'''(0)$, the general boundary conditions are imposed, that is, we assume the beam is restrained at the ends $x = 0$ and $x = L$ by the linear and torsional springs. Substituting eqns (4) and (5) of boundary conditions into eqn (26), we can get the following

$$\begin{aligned}\phi(x) &= R_1(\lambda x)\phi(0) + \frac{1}{\lambda}R_2(\lambda x)\phi'(0) + \frac{\beta_1}{EI\lambda^2}R_3(\lambda x)\phi'(0) \\ &\quad - \frac{K_3}{EI\lambda^3}R_4(\lambda x)\phi(0) \\ &\quad + \frac{1}{EI\lambda^3}\sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)R_4[\lambda(x-x_i)]u(x-x_i) \\ &= \phi(0)\left[R_1(\lambda x) - \frac{K_3}{EI\lambda^3}R_4(\lambda x)\right] \\ &\quad + \phi'(0)\left[\frac{1}{\lambda}R_2(\lambda x) + \frac{\beta_1}{EI\lambda^2}R_3(\lambda x)\right] \\ &\quad + \frac{1}{EI\lambda^3}\sum_{i=1}^{2} K_i\phi(x_i)B_iR_4[\lambda(x-x_i)]u(x-x_i) \\ &= e_1S_1(\lambda x) + e_2S_2(\lambda x) \\ &\quad + \frac{1}{EI\lambda^3}\sum_{i=1}^{2} K_i\phi(x_i)B_iR_4[\lambda(x-x_i)]u(x-x_i)\end{aligned} \tag{27}$$

where

$$e_1 = \phi(0) \tag{28}$$

$$e_2 = \phi'(0) \tag{29}$$

$$S_1(\lambda x) = R_1(\lambda x) - \frac{K_3}{EI\lambda^3}R_4(\lambda x) \tag{30}$$

$$S_2(\lambda x) = \frac{1}{\lambda} R_2(\lambda x) + \frac{\beta_1}{EI\lambda^3} R_3(\lambda x). \tag{31}$$

Substituting eqn (27) into eqn (6) of the boundary conditions gives

$$\begin{aligned}
&e_1\lambda^2\left[R_3(\lambda L) - \frac{K_3}{EI\lambda^3} R_2(\lambda L)\right] + e_2\lambda^2\left[\frac{1}{\lambda} R_4(\lambda L) + \frac{\beta_1}{EI\lambda^2} R_1(\lambda L)\right] \\
&\quad + \frac{1}{EI\lambda}\sum_{i=1}^{2} K_i\phi(x_i)B_iR_2[\lambda(x - x_i)] \\
&= \frac{-\beta_2}{EI}\left\{e_1\lambda\left[R_4(\lambda L) - \frac{K_3}{EI\lambda^3} R_3(\lambda L)\right] + e_2\lambda\left[\frac{1}{\lambda} R_1(\lambda L) + \frac{\beta_1}{EI\lambda^2} R_2(\lambda L)\right]\right. \\
&\quad \left. + \frac{1}{EI\lambda^2}\sum_{i=1}^{2} K_i\phi(x_i)B_iR_3[\lambda(L - x_i)]\right\}.
\end{aligned} \tag{32}$$

Equation (32) can be simplified further as

$$e_1F_1 + e_2F_2 = \sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)\left\{\frac{-1}{EI\lambda} R_2[\lambda(L - x_i)] + \frac{-\beta_2}{E^2I^2\lambda^2} R_3[\lambda(L - x_i)]\right\} \tag{33}$$

where

$$F_1 = \left(\lambda^2 - \frac{\beta_2 K_3}{E^2I^2\lambda^2}\right) R_3(\lambda L) - \frac{K_3}{EI\lambda} R_2(\lambda L) + \frac{\beta_2\lambda}{EI} R_4(\lambda L) \tag{34}$$

$$F_2 = \left(\frac{\beta_1 + \beta_2}{EI}\right) R_1(\lambda L) - \frac{\beta_1\beta_2}{E^2I^2\lambda} R_2(\lambda L) + \lambda R_4(\lambda L). \tag{35}$$

Similarly, substituting eqn (27) into eqn (7) of the boundary condition gives the following

$$e_1F_3 + e_2F_4 = \sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)\left\{\frac{K_4}{E^2I^2\lambda^3} R_4[\lambda(L - x_i)] + \frac{1}{EI} R_1[\lambda(L - x_i)]\right\} \tag{36}$$

where

$$F_3 = \lambda^3 R_2(\lambda L) - \frac{K_3}{EI} R_1(\lambda L) - \frac{K_4}{EI} S_1(\lambda L) \tag{37}$$

$$F_4 = \lambda^2 R_3(\lambda L) + \frac{\beta_1\lambda}{EI} R_4(\lambda L) - \frac{K_4}{EI} S_2(\lambda L). \tag{38}$$

Solving eqn (33) and eqn (36) for e_1 and e_2, we can get the following

$$\begin{aligned}
e_1 = \sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)&\left\{\frac{-F_4}{EI\lambda} R_2[\lambda(L - x_i)] - \frac{\beta_2F_4}{E^2I^2\lambda^2} R_3[\lambda(L - x_i)]\right. \\
&\left. - \frac{K_4F_2}{E^2I^2\lambda^3} R_4[\lambda(L - x_i)] + \frac{F_2}{EI} R_1[\lambda(L - x_i)]\right\}/(F_1F_4 - F_2F_3)
\end{aligned} \tag{39}$$

$$e_2 = \sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)\left\{\frac{-F_3}{EI\lambda}R_2[\lambda(L-x_i)] - \frac{\beta_2 F_3}{E^2I^2\lambda^2}R_3[\lambda(L-x_i)] - \frac{K_4F_1}{E^2I^2\lambda^3}R_4[\lambda(L-x_i)] + \frac{F_1}{EI}R_1[\lambda(L-x_i)]\right\}/(F_2F_3-F_1F_4). \quad (40)$$

Setting $x = x_1$ and $x = x_2$ in eqn (27), we can obtain

$$\phi(x_1) = D_1 \sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)[S_1(\lambda x_1)T_{1xi} - S_2(\lambda x_1)T_{2xi}] \quad (41)$$

$$\phi(x_2) = D_1 \sum_{i=1}^{2} K_i\phi(x_i)B_i(\lambda)[S_1(\lambda x_2)T_{1xi} - S_2(\lambda x_2)T_{2xi}] + \frac{1}{EI\lambda^3}K_1\phi(x_1)B_1R_4[\lambda(x_2-x_1)] \quad (42)$$

where

$$T_{1xi} = \frac{-F_4}{EI\lambda}R_2[\lambda(L-x_i)] - \frac{\beta_2 F_4}{E^2I^2\lambda^2}R_3[\lambda(L-x_i)] - \frac{K_4F_2}{E^2I^2\lambda^3}R_4[\lambda(L-x_i)] + \frac{F_2}{EI}R_1[\lambda(L-x_i)] \quad i = 1,2 \quad (43)$$

$$T_{2xi} = \frac{-F_3}{EI\lambda}R_2[\lambda(L-x_i)] - \frac{\beta_2 F_3}{E^2I^2\lambda^2}R_3[\lambda(L-x_i)] - \frac{K_4F_1}{E^2I^2\lambda^3}R_4[\lambda(L-x_i)] + \frac{F_1}{EI}R_1[\lambda(L-x_i)] \quad i = 1,2 \quad (44)$$

$$D_1 = \frac{1}{F_1F_4 - F_2F_3}. \quad (45)$$

From eqns (41) and (42) we come up with two homogeneous equations as follows

$$c_{11}(\lambda)\phi(x_1) + c_{12}(\lambda)\phi(x_2) = 0 \quad (46)$$

$$c_{21}(\lambda)\phi(x_1) + c_{22}(\lambda)\phi(x_2) = 0 \quad (47)$$

where

$$c_{11} = D_1S_1(\lambda x_1)K_1B_1T_{1x1} - D_1S_2(\lambda x_1)K_1B_1T_{2x1} - 1 \quad (48)$$

$$c_{12} = D_1S_1(\lambda x_1)K_2B_2T_{1x2} - D_1S_2(\lambda x_1)K_2B_2T_{2x2} \quad (49)$$

$$c_{21} = D_1S_1(\lambda x_2)K_1B_1T_{1x1} - D_1S_2(\lambda x_2)K_1B_1T_{2x1} + \frac{1}{EI\lambda^3}K_1B_1R_4[\lambda(x_2-x_1)] \quad (50)$$

$$c_{22} = D_1S_1(\lambda x_2)K_2B_2T_{1x2} - D_1S_2(\lambda x_2)K_2B_2T_{2x2} - 1. \quad (51)$$

The eigenvalues λ_n and further the natural frequencies ω_n are determined by setting the determinant of the coefficients in eqns (46) and (47) to be zero. Thus

$$c_{11}(\lambda_n)c_{22}(\lambda_n) - c_{12}(\lambda_n)c_{21}(\lambda_n) = 0. \quad (52)$$

From eqn (46), we can write

$$\phi(x_1) = -\frac{c_{12}(\lambda_n)}{c_{11}(\lambda_n)}\phi(x_2)$$

substituting it into eqn (27) and dividing by $\phi(x_2)$, we can obtain the mode shapes of the beam as follows

$$\begin{aligned}\phi(x) = {} & D_1 S_1(\lambda_n x) K_1 B_1(\lambda_n) T_{1x1}\left[\frac{-c_{12}(\lambda_n)}{c_{11}(\lambda_n)}\right] + D_1 S_1(\lambda_n x) K_2 B_2 T_{1x2} \\ & - D_1 S_2(\lambda_n x) K_1 B_1(\lambda_n) T_{2x1}\left[\frac{-c_{12}(\lambda_n)}{c_{11}(\lambda_n)}\right] + D_1 S_2(\lambda_n x) K_2 B_2 T_{2x2} \\ & + \frac{1}{EI\lambda^3}\left[\frac{-c_{12}(\lambda_n)}{c_{11}(\lambda_n)}\right] K_1 B_1(\lambda_n) R_4[\lambda_n(x-x_1)]u(x-x_1) \\ & + \frac{1}{EI\lambda^3} K_2 B_2(\lambda_n) R_4[\lambda_n(x-x_2)]u(x-x_2).\end{aligned} \tag{53}$$

4. FORCED VIBRATION ANALYSIS

For the forced vibration of the damped system, expanding $y_b(x, t)$ in terms of the undamped system eigenfunctions yields

$$y_b(x,t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t) \tag{54}$$

where the $\phi_i(x)$ are the eigenfunctions of the beam, $q_i(t)$ are unknown and time dependent generalized co-ordinates. Substituting the series solution into eqn (17) and assume the following

$$F(x,t) = \rho A \sum_{i=1}^{\infty} f_i(t)\phi_i(x) \tag{55}$$

where

$$f_i(t) = \frac{1}{\mu_i}\int F(x,t)\phi_i(x)\,\mathrm{d}x. \tag{56}$$

Then eqn (17) becomes

$$\begin{aligned}\sum_{i=1}^{\infty} \{ & EI\phi''''_i(x)q_i(t) + 2\rho A\zeta\omega_i\phi_i(x)\dot{q}_i(t) + \rho A\phi_i(x)\ddot{q}_i(t) - \rho A f_i(t)\phi_i(x) \\ & - [K_1 B_1\phi_i(x)q_i(t)\,\delta(x-x_1) + K_2 B_2\phi_i(x)q_i(t)\,\delta(x-x_2)]\}.\end{aligned} \tag{57}$$

The following equation can be achieved by using the previous discussion of free vibration analysis.

$$EI\phi''''_i(x) = \rho A\omega_i^2\phi_i(x) + [K_1 B_1\phi_i(x)\,\delta(x-x_1) + K_2 B_2\phi_i(x)\,\delta(x-x_2)]. \tag{58}$$

Substituting eqn (58) into eqn (57) gives

$$\sum_{i=1}^{\infty}[\ddot{q}_i(t)+2\zeta_i\omega_i\dot{q}_i(t)-\omega_i^2 q_i(t)-f_i(t)]\rho A\phi_i(x) = 0. \tag{59}$$

Multiply the above equation by ϕ_j and integrate from 0 to L, we can obtain

$$\sum_{i=1}^{\infty}[\ddot{q}_i(t)+2\zeta_i\omega_i\dot{q}_i(t)+\omega_i^2 q_i(t)-f_i(t)]\int_0^L \rho A\phi_i\phi_j\,\mathrm{d}x = 0. \tag{60}$$

At this stage, it is necessary to derive the orthogonal property of the eigenfunctions $\phi_i(x)$ so that the time dependent generalized coordinates, $q_i(t)$, can be solved. To do so, the governing equation and boundary conditions must be adopted and manipulated carefully in the following.

$$(EI\phi'')''-\rho A\omega^2\phi = K_1B_1\phi\delta(x-x_1)+K_2B_2\phi\delta(x-x_2) \tag{61}$$

$$(EI\phi'')'+K_3\phi = 0|_{x=0} \tag{62}$$

$$(EI\phi'')'-K_4\phi = 0|_{x=L} \tag{63}$$

$$EI\phi''-\beta_1\phi' = 0|_{x=0} \tag{64}$$

$$EI\phi''+\beta_2\phi' = 0|_{x=L}. \tag{65}$$

Equation (61) can be written for ϕ_i and ϕ_j, individually. Multiply the first equation by ϕ_j and the second equation by ϕ_i, then integrate from 0 to L we can acquire the following equation

$$\int_0^L \phi_j(EI\phi_i'')''\,\mathrm{d}x-\int_0^L \phi_j[K_1B_1\phi_i\delta(x-x_1)+K_2B_2\phi_i\delta(x-x_2)]\,\mathrm{d}x=\omega_i^2\int_0^L \rho A\phi_i\phi_j\,\mathrm{d}x \tag{66}$$

$$\int_0^L \phi_i(EI\phi_j'')''\,\mathrm{d}x-\int_0^L \phi_i[K_1B_1\phi_j\delta(x-x_1)+K_2B_2\phi_j\delta(x-x_2)]\,\mathrm{d}x=\omega_j^2\int_0^L \rho A\phi_i\phi_j\,\mathrm{d}x. \tag{67}$$

Subtracting eqn (66) from eqn (67), we can obtain

$$(\omega_i^2-\omega_j^2)\int_0^L \rho A\phi_i\phi_j\,\mathrm{d}x = \int_0^L \phi_j(EI\phi_i'')''\,\mathrm{d}x-\int_0^L \phi_i(EI\phi_j'')''\,\mathrm{d}x. \tag{68}$$

After some algebra, the right-hand side of eqn (68) turns out to be zero, so eqn (68) can be written as follows

$$(\omega_i^2-\omega_j^2)\int_0^L \rho A\phi_i\phi_j\,\mathrm{d}x = 0. \tag{69}$$

Finally, the following orthogonality condition can be achieved when the natural frequency ω_i is different from ω_j.

$$\int_0^L \rho A\phi_i\phi_j\,\mathrm{d}x = 0 = \mu_i\delta_{ij} \tag{70}$$

where μ_i is the generalized mass and δ_{ij} is the Kronecker delta function. Once we prove the existence of the orthogonal eigenfunction of the beam, we can obtain a differential equation involving q_i as follows by substituting eqn (70) into eqn (60)

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{Q_i}{\mu_i} \tag{71}$$

where $Q_i(t)$ is the generalized force and μ_i is the generalized mass. They are defined as follows

$$Q_i(t) = \int_0^L F(x,t)\phi_i(x)\,\mathrm{d}x \tag{72}$$

$$\mu_i = \rho A \int_0^L \phi_i^2\,\mathrm{d}x. \tag{73}$$

Solving the differential eqn (71) results in the following expression to the generalized co-ordinates:

$$q_i(t) = \mathrm{e}^{-\zeta_i\omega_i t}\left[\frac{q(0)\cos(\omega_{di}t-\varphi_i)}{\sqrt{1-\zeta_i^2}} + \frac{\dot{q}(0)}{\omega_{di}}\sin\omega_{di}t\right] + \frac{1}{\mu_i\omega_{di}}\int_0^t \mathrm{e}^{-\zeta_i\omega_i(t-\tau)}Q_i(\tau)\sin\omega_{di}(t-\tau)\,\mathrm{d}\tau \tag{74}$$

where

$$\omega_{di} = \omega_i\sqrt{1-\zeta_i^2}\varphi_i = \tan^{-1}\frac{\zeta_i}{\sqrt{1-\zeta_i^2}}.$$

Once the time dependent generalized coordinates q_i(t) is evaluated, it is quite straightforward to calculate the forced vibrational response of the beam by using eqn (54).

5. RANDOM VIBRATION ANALYSIS

The theory of random vibration can be used to solve a very large class of engineering problems since random excitation can be a satisfactory approximate model for a wide range of real excitations. In this study, stationary random excitation and zero initial conditions are assumed, and only the stationary random response is considered. From eqn (71) the equation for the generalized displacement due to the external force can be written as

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{Q_i}{\mu_i}. \tag{71}$$

The generalized displacement in terms of the system transfer function is given as

$$q_i(t) = \frac{1}{\mu_i}\int_{-\infty}^{+\infty} H_i(\omega)\exp(i\omega t)Q_i(\omega)\,\mathrm{d}\omega \tag{75}$$

where $H_i(\omega)$ is the frequency response function, which can be expressed as

$$H_i(\omega) = [\omega_i^2 + 2i\zeta_i\omega_i\omega - \omega^2]^{-1}. \tag{76}$$

Since the random excitation $F(x, t)$ is assumed to be stationary in time, then so is the generalized force $Q_i(t)$, and the cross-correlation function between the response at x_1 and x_2 can be written as

$$R_{y_{bx1}y_{bx2}}(x_1, x_2, \tau) = E[y_b(x_1, t)y_b(x_2, t+\tau)]$$
$$= E\left[\sum_{m=1}^{\infty} \phi_m(x_1)q_m(t) \sum_{n=1}^{\infty} \phi_n(x_2)q_n(t+\tau)\right] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_m(x_1)\phi_n(x_2)R_{q_mq_n}(\tau). \quad (77)$$

Now, using eqns (75) and (77), one obtains

$$R_{y_{bx1}y_{bx2}}(x_1, x_2, \tau) = \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_m(x_1)\phi_n(x_2)\int_{-\infty}^{-\infty} H_m(\omega)H_n^*(\omega)S_{Q_mQ_n}(\omega)\exp(i\omega\tau)\,d\omega \quad (78)$$

where $S_{Q_mQ_n}(\omega)$ is the cross-spectral density function between $Q_m(t)$ and $Q_n(t)$, which can be obtained explicitly through double integration from $S_{F_{x1}F_{x2}}(x_1, x_2, \omega)$ which is the distributed cross-spectral density function between the random excitation $F(x_1, t)$ and $F(x_2, t)$ and $H_n^*(\omega)$ is the complex conjugate of $H_n(\omega)$. Therefore, if $S_{F_{x1}F_{x2}}(x_1, x_2, \omega)$ is given, $R_{y_{bx1}y_{bx2}}(x_1, x_2, \tau)$ can be computed easily from eqn (78). For $x_1 = x_2 = x$, the response cross-correlation function reduces to the autocorrelation function

$$R_{y_b}(x, \tau) = \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_m(x)\phi_n(x)\int_{-\infty}^{+\infty} H_m(\omega)H_n^*(\omega)S_{Q_mQ_n}(\omega)\exp(i\omega\tau)\,d\omega. \quad (79)$$

Upon letting $\tau = 0$, the mean square value of the displacement at the point x is obtained as

$$E[y_b^2(x)] = \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_m(x)\phi_n(x)\int_{-\infty}^{+\infty} H_m(\omega)H_n^*(\omega)S_{Q_mQ_n}(\omega)\,d\omega. \quad (80)$$

The cross-spectral density function $S_{F_{x1}F_{x2}}(x_1, x_2, \omega)$ of the distributed random excitation is assumed to be $S_0\exp(-\alpha|x_1 - x_2|)\exp(-\beta\omega^2)$, then the spectral density function of the generalized force can be written as

$$S_{Q_mQ_n}(\omega) = \frac{1}{\mu_m\mu_n}\int_0^L\int_0^L \phi_m(x_1)\phi_n(x_2)S_0\exp(-\alpha|x_1 - x_2|)\exp(-\beta\omega^2)\,dx_1\,dx_2. \quad (81)$$

For simplicity, one can define

$$S_{mn} = S_0\int_0^L\int_0^L \phi_m(x_1)\phi_n(x_2)\exp(-\alpha|x_1 - x_2|)\,dx_1\,dx_2. \quad (82)$$

From eqn (80), the mean square value of the displacement can be written as

$$E[y_b^2(x)] = \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [\phi_m(x)\phi_n(x)S_{mn}/\mu_m\mu_n]\int_{-\infty}^{+\infty} H_m(\omega)H_n^*(\omega)\exp(-\beta\omega^2)\,d\omega. \quad (83)$$

6. NUMERICAL SOLUTIONS AND DISCUSSIONS

6.1. *Free vibration analysis*

We adopt the bisection method to solve the frequency eqn (52), then we can obtain the natural frequency ω_n based on the relation $\lambda^4 = \rho A\omega^2/EI$. Substituting ω_n into eqn (53),

Table 1. Natural frequencies of present study and exact solution

Mode	ω (rad s^{-1}) Present study (simply-supported)	ω (rad s^{-1}) Exact solution (simply-supported)	ω (rad s^{-1}) Present study (clamped)	ω (rad s^{-1}) Exact solution (clamped)
1	449	449.1	1017.9	1018.1
2	1796.02	1796.5	2805.9	2806.4
3	4042	4042.1	5501.6	5501.7
4	7184.5	7185.9	9093.3	9094.7
5	11,226.8	11,228.1	13,583.8	13,585.9

we can get the numerical solution of the beam mode shapes. In the present study, we assume the material properties and dimension sizes of the beam and the spring constants of the spring–mass system as follows:

$$E = 200\times10^9\ \mathrm{N/m^{-2}},\ \rho_b = 7860\ \mathrm{Kg\,m^{-3}},\ L_b = 4\ \mathrm{m},\ w_b = 0.3\ \mathrm{m},$$

$$H_b = 0.5\ \mathrm{m},\ I_b = \tfrac{1}{12}w_bH_b^3,\ L_1 = 2\ \mathrm{m},\ H_1 = 0.5\ \mathrm{m},\ w_1 = 0.3\ \mathrm{m},$$

$$I_1 = \tfrac{1}{12}m_1(L_1^2+H_1^2),\ d_1 = d_2 = 0.6\ \mathrm{m},\ x_1 = 1.4\ \mathrm{m},\ x_2 = 2.6\ \mathrm{m}$$

$$K_1 = K_2 = 1\times10^{10}\ \mathrm{N\,m^{-1}},\ \zeta_i = 0.05.$$

First of all, we must check the accuracy of the proposed method before we start analyzing the other special cases or general case. We compare the results of the special cases of the present study with $m_1 = 0.1$ kg with the exact solution with $m_1 = 0$. Then the results of comparisons are presented in Table 1 as follows

We can accept the accuracy of the numerical results of the present study because they are very close to the exact solution, therefore, we use the same numerical method to analyze some different cases in the following.

6.1.1. *Special case.*

(i) The beam is assumed as simply-supported at both ends. Theoretically speaking, $K_3 \to \infty$, $K_4 \to \infty$ and $\beta_1 = \beta_2 = 0$ must be adopted in this particular case, however, since we are working on the numerical solution, let us assume $K_3 = K_4 = 1\times10^{20}$ N m^{-1} and $\beta_1 = \beta_2 = 1\times10^{-10}$ N m^{-1}, $m_1 = 200$ kg. Then the numerical results are calculated and presented in Table 2 and Fig. 2 separately.

(ii) The beam is assumed as clamped at both ends. In order to obtain the numerical solutions in this case, we assume $K_3 = K_4 = 1\times10^{20}$ N m^{-1}, $\beta_1 = \beta_2 = 1\times10^{20}$ N m^{-1}, and the other parameter values are the same as those in case (a). Then the numerical results are calculated and presented in Table 3 and Fig. 3, separately.

6.1.2. *General case.*

(i) We assume $K_3 = K_4 = 1\times10^{10}$ N m^{-1}, $\beta_1 = \beta_2 = 1\times10^{10}$ N m^{-1} and the other parameter values are exactly the same as those in special case except m_1 is changed to 500 kg. The following numerical results are achieved and presented in Table 4 and Fig. 4 based on the above parameter values

Table 2. First five natural frequencies of the simply-supported beam

Mode	λ	ω (rad s^{-1})
1	0.772781	434
2	1.549144	1747
3	2.354843	4037
4	3.048975	6768
5	3.642023	9657

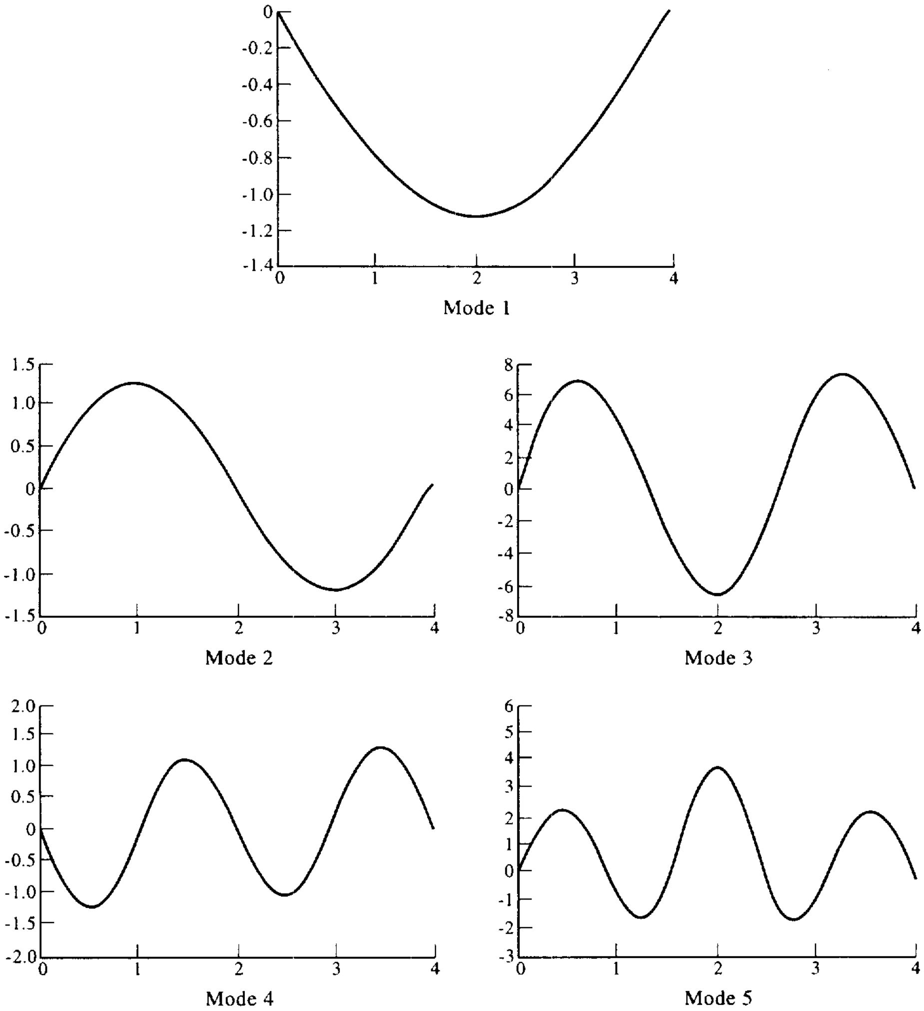

Fig. 2. First five mode shapes of the simply-supported beam.

Table 3. First five natural frequencies of the clamped beam

Mode	λ	ω (rad s^{-1})
1	1.1622224	983
2	1.9236416	2694
3	2.7482261	5499
4	3.4044699	8438
5	3.6967433	9949

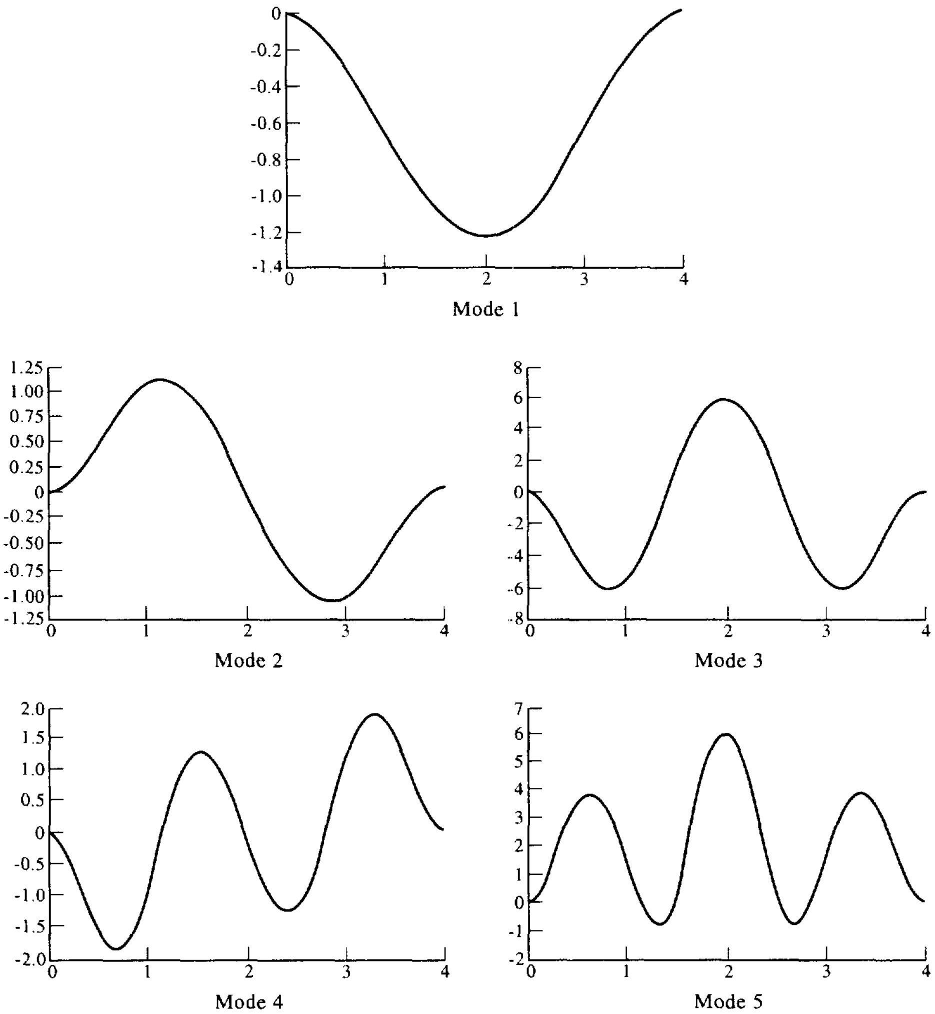

Fig. 3. First five mode shapes of the clamped beam.

Table 4. First five natural frequencies of the general restrained beam

Mode	λ	ω (rad s^{-1})
1	1.0628212	822
2	1.6554582	1995
3	2.1879959	3485
4	2.532351	4669
5	2.9527878	6348

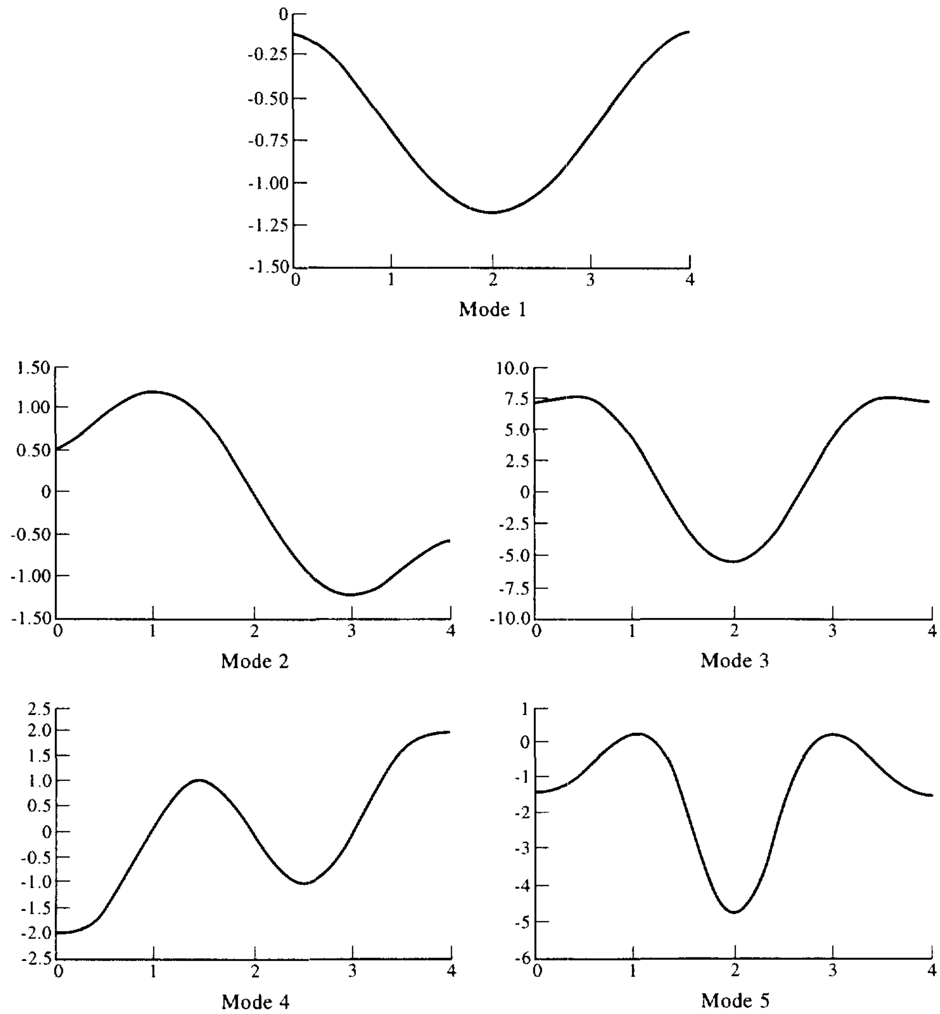

Fig. 4. First five mode shapes of the general restrained beam.

Table 5. First five natural frequencies of the general restrained beam

Mode	λ	ω (rad s^{-1})
1	1.1053936	890
2	1.7095568	2128
3	2.1912832	3496
4	2.6668199	5178
5	3.26543	7763

(ii) When we design a machine structure, we can always change some parameter to alter and improve the character of the primary structure. Now we replace the mass of the spring–mass system above the beam from 500 to 10 kg and we can get Table 5 and Fig. 5 as follows.

(iii) Theoretically, we can reduce the mass of the spring–mass system to increase the fundamental natural frequency. Similarly, we can also reduce the mass moment of inertia to increase the lowest natural frequency when the other parameter values are exactly the same and we can change the spring constant of K_1 and K_2 to get the different fundamental natural frequency. After some calculations and computer drawing we can obtain Figs 6–8 in the following. It can be found that the fundamental natural frequency gets smaller as the mass or mass moment of inertia of the spring–mass system gets larger.

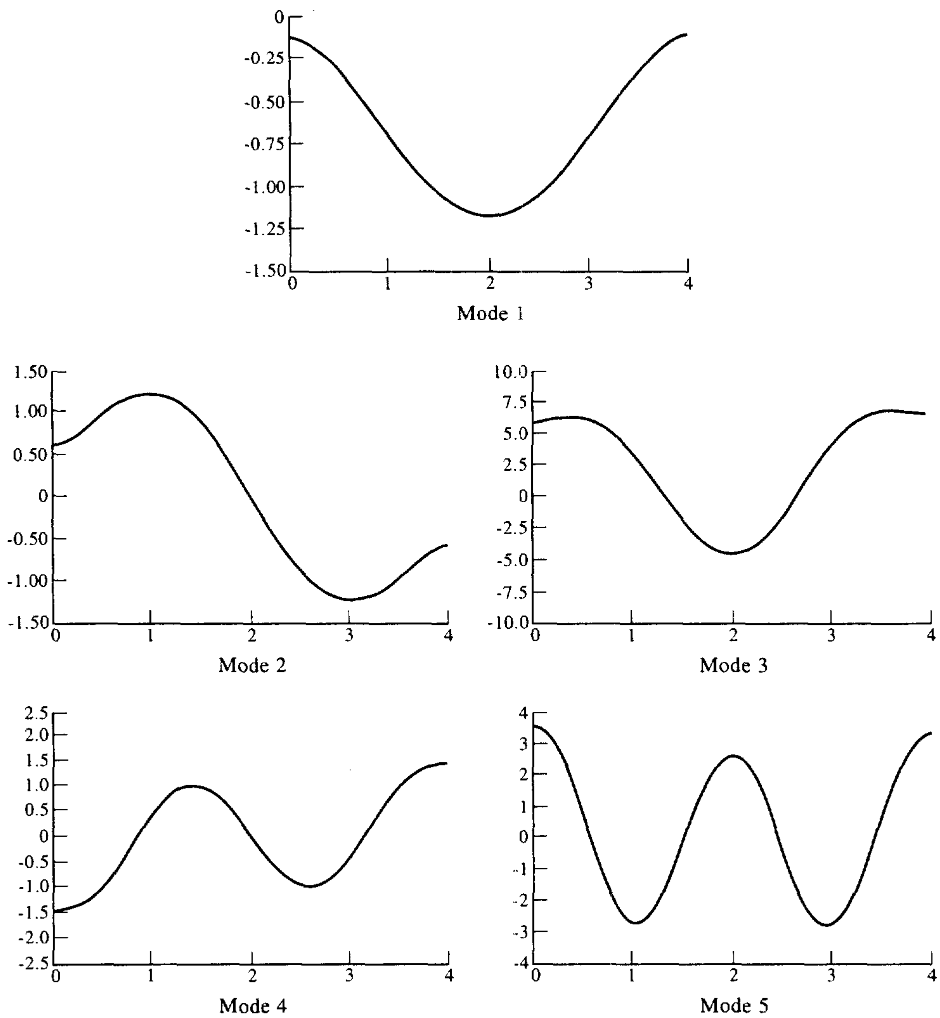

Fig. 5. First five mode shapes of the general restrained beam.

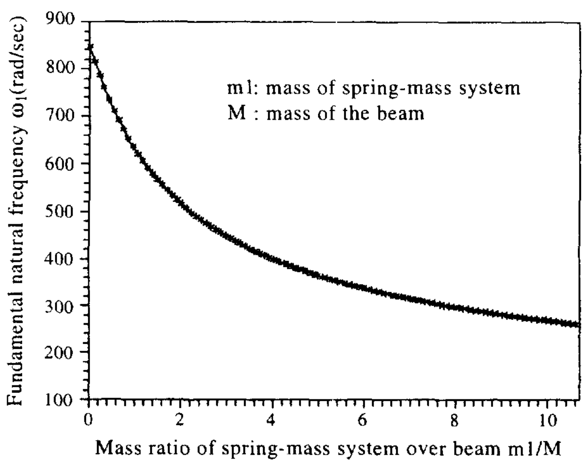

Fig. 6. Fundamental natural frequency of the general restrained beam for various mass of spring–mass system.

6.2. *Forced vibration analysis*

6.2.1. *Special case.* It is quite understood that different boundary conditions will result in different dynamic response. Suppose that the boundary conditions of the beam are assumed as simply-supported and clamped–clamped, respectively, and the beam is subjected

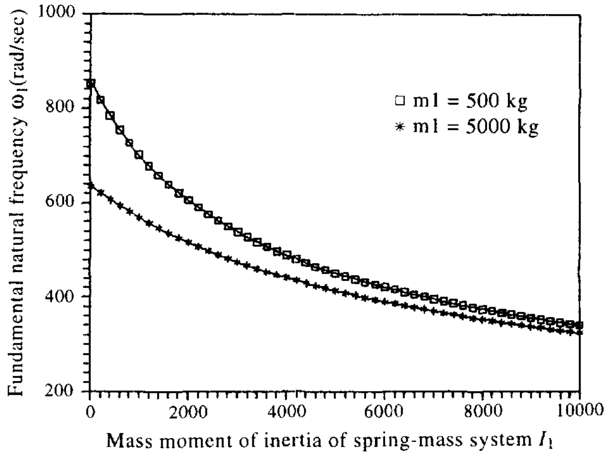

Fig. 7. Fundamental natural frequency of the general restrained beam for various mass moment of inertia of spring-mass system.

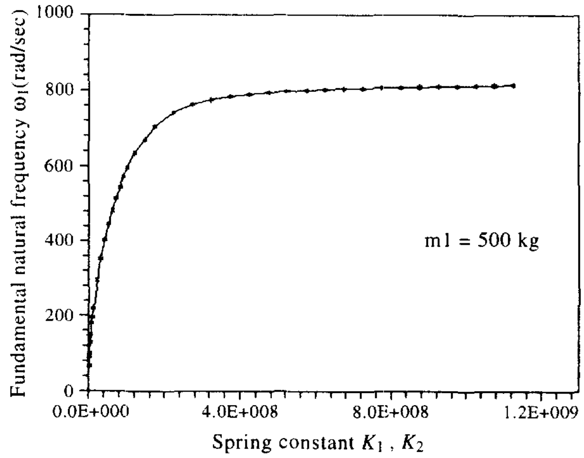

Fig. 8. Fundamental natural frequency of the general restrained beam for various spring constant K_1, K_2.

to a uniform distributed force $F(x, t) = 5 \times 10^5 \sin \Omega t$ N m^{-1}, where $\Omega = 10$ rad s^{-1}, after some numerical calculations, the dynamic response of the beam are presented in Figs 9 and 10, separately.

6.2.2. *General case.*

(i) In this general case, $F(x, t) = 10^7 \times \sin \Omega t$ N m^{-1}, where $\Omega = 10$ rad s^{-1} is adopted. The dynamic response of the general restrained beam is presented in Figs 11 and 12, respectively, for different mass of spring–mass system. As it can be detected from the figures, the dynamic response of the beam is greater if the mass of the spring–mass system is larger.

(ii) The dynamic response of the general restrained beam is presented in Figs 13 and 14, respectively, for $F(x, t) = 10^7 \times \sin \Omega t$ N m^{-1} ($\Omega = 1000$ rad s^{-1}) and different driving frequency. As can be seen from Fig. 13, the dynamic response of the beam is smaller if the mass of the spring–mass system is larger. It is quite interesting to note that the result of Fig. 13 is different from that of Fig. 11, since now the driving frequency of the excitation is closer to the fundamental natural frequency of the beam with the smaller mass of spring–mass system. In Fig. 14, whenever the forcing frequency is close to one of the natural frequencies, the dynamic response of the beam becomes larger, which is quite reasonable.

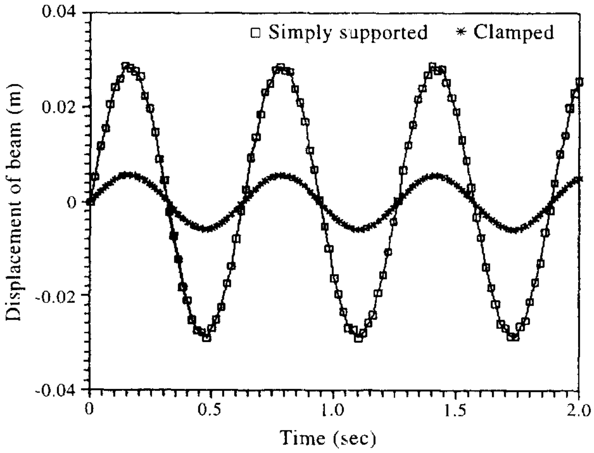

Fig. 9. Dynamic response of the mid point of the beam.

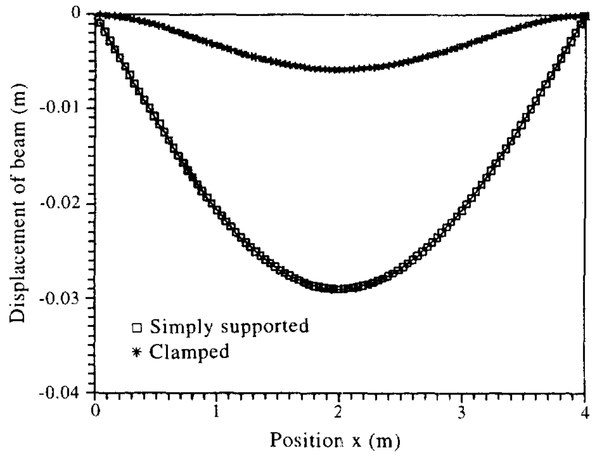

Fig. 10. Dynamic response of the beam at $t = 1.1$ s.

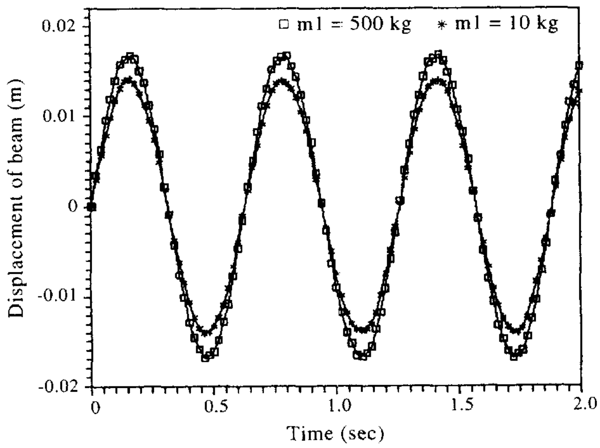

Fig. 11. Dynamic response of the mid point of the general restrained beam ($\Omega = 10\,\text{rad}\,\text{s}^{-1}$).

6.3. *Random vibration analysis*

Based on the proposed random vibration analysis described previously, the mean square value of the displacement of the beam can be determined from eqn (83). In the numerical computation, the cross-spectral density function $S_{F_{x1}F_{x2}}(x_1, x_2, \omega)$ of the distributed random excitation is assumed to be $S_0(-\alpha|x_1 - x_2|)\exp(-\beta\omega^2)$, where $\alpha = \beta = 1$

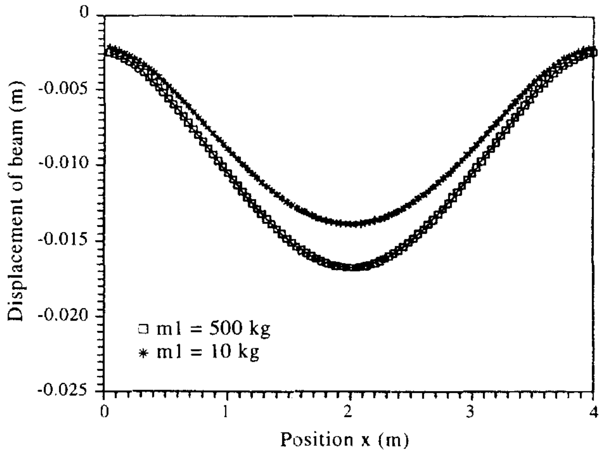

Fig. 12. Dynamic response of the general restrained beam at $t = 1.1$ s.

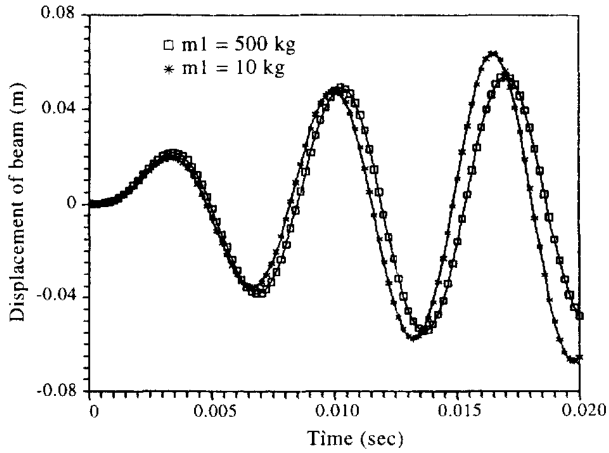

Fig. 13. Dynamic response of the mid point of the general restrained beam ($\Omega = 1000 \text{ rad s}^{-1}$).

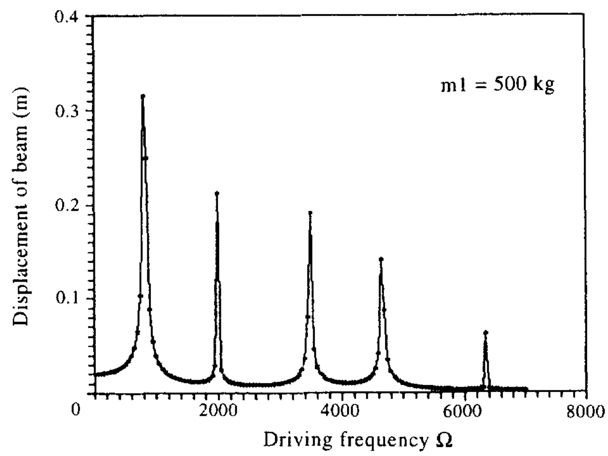

Fig. 14. Dynamic response of the beam for different excitation frequency.

and $S_0 = 1 \times 10^{11}$ N m^{-2} s, and the calculated mean square value of the displacement of the beam is shown in Fig. 15.

7. CONCLUSIONS

In this paper, the natural frequencies and mode shapes of a Bernoulli–Euler beam with a two degree-of-freedom spring–mass system are determined by using Laplace transform

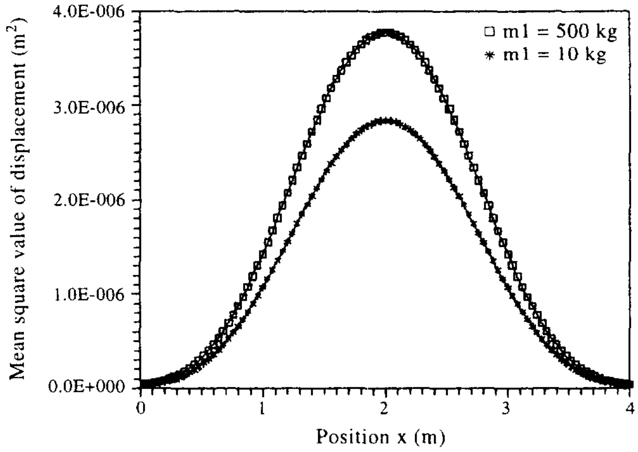

Fig. 15. Mean square value of the displacement of the beam.

with respect to the spatial variable and the deterministic and random vibration responses of the beam are obtained by using modal analysis. The various spring constants of the boundary conditions of the beam are considered to model those special structures and machines, also different parameters of the spring–mass system are taken into consideration to study the fundamental natural frequencies of the beam. It should be emphasized that the proposed analysis is quite important for the design of some components of the buildings or machine tools.

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